

# The Annihilating-Ideal Graph of a Ring

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## Abstract

Let  $S$  be a semigroup with 0 and  $R$  be a ring with 1. We extend the definition of the zero-divisor graphs of commutative semigroups to not necessarily commutative semigroups. We define an annihilating-ideal graph of a ring as a special type of zero-divisor graph of a semigroup. We introduce two ways to define the zero-divisor graphs of semigroups. The first definition gives a directed graph  $\Gamma(S)$ , and the other definition yields an undirected graph  $\overline{\Gamma}(S)$ . It is shown that  $\Gamma(S)$  is not necessarily connected, but  $\overline{\Gamma}(S)$  is always connected and  $\text{diam}(\overline{\Gamma}(S)) \leq 3$ . For a ring  $R$  define a directed graph  $\mathbb{A}\text{POG}(R)$  to be equal to  $\Gamma(\mathbb{I}\text{PO}(R))$ , where  $\mathbb{I}\text{PO}(R)$  is a semigroup consisting of all products of two one-sided ideals of  $R$ , and define an undirected graph  $\overline{\mathbb{A}\text{POG}}(R)$  to be equal to  $\overline{\Gamma}(\mathbb{I}\text{PO}(R))$ . We show that  $R$  is an Artinian (resp., Noetherian) ring if and only if  $\mathbb{A}\text{POG}(R)$  has DCC (resp., ACC) on some special subset of its vertices. Also, It is shown that  $\overline{\mathbb{A}\text{POG}}(R)$  is a complete graph if and only if either  $(D(R))^2 = 0$ ,  $R$  is a direct product of two division rings, or  $R$  is a local ring with maximal ideal  $\mathfrak{m}$  such that  $\mathbb{I}\text{PO}(R) = \{0, \mathfrak{m}, \mathfrak{m}^2, R\}$ . Finally, we investigate the diameter and the girth of square matrix rings over commutative rings  $M_{n \times n}(R)$  where  $n \geq 2$ .

**Key Words:** Rings; Semigroups; Zero-Divisor Graphs; Annihilating-Ideal Graphs.

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## 1 introduction

In [11], I. Beck associated to a commutative ring  $R$  its zero-divisor graph  $G(R)$  whose vertices are the zero-divisors of  $R$  (including 0), and two distinct vertices  $a$  and  $b$  are adjacent if  $ab = 0$ . In [9], Anderson and Livingston introduced and studied the subgraph  $\Gamma(R)$  (of  $G(R)$ ) whose vertices are the nonzero zero-divisors of  $R$ . This graph turns out to best exhibit the properties of the set of zero-divisors of  $R$ , and the ideas and problems introduced in [9] were further studied in [4, 8, 10]. In [20], Redmond extended the definition of zero-divisor graph to non-commutative rings. Some fundamental results concerning zero-divisor graph for a non-commutative ring were given in [5, 6, 21]. For a commutative ring  $R$  with 1, denoted by  $\mathbb{A}(R)$ , the set of ideals with nonzero annihilator. The annihilating-ideal graph of  $R$  is an undirected graph  $\mathbb{A}\mathbb{G}(R)$  with vertices  $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{0\}$ , where distinct vertices  $I$  and  $J$  are adjacent if  $IJ = (0)$ . The concept of the annihilating-ideal graph of a commutative ring was introduced in [12, 13]. Several fundamental results concerning  $\mathbb{A}\mathbb{G}(R)$  for a commutative ring were given in [1, 2, 3, 7]. For a ring  $R$ , let  $D(R)$  be the set of one-sided zero-divisors of  $R$  and  $\mathbb{I}\text{PO}(R) = \{A \subseteq R : A = IJ \text{ where } I \text{ and } J \text{ are left or right ideals of } R\}$ . Let  $S$  be a semigroup with 0, and

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$D(S)$  be the set of one-sided zero-divisors of  $S$ . The zero-divisor graph of a commutative semigroup is an undirected graph with vertices  $Z(S)^*$  (the set of non-zero zero-divisors) and two distinct vertices  $a$  and  $b$  are adjacent if  $ab = 0$ . The zero-divisor graph of a commutative semigroup was introduced in [16] and further studied in [14, 22, 23, 24].

Let  $\Gamma$  be a graph. For vertices  $x$  and  $y$  of  $\Gamma$ , let  $d(x, y)$  be the length of a shortest path from  $x$  to  $y$  ( $d(x, x) = 0$  and  $d(x, y) = \infty$  if there is no such path). The diameter of  $\Gamma$  is defined as  $\text{diam}(\Gamma) = \sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } \Gamma\}$ . The girth of  $\Gamma$ , denoted by  $\text{gr}(\Gamma)$ , is the length of a shortest cycle in  $\Gamma$  ( $\text{gr}(\Gamma) = \infty$  if  $\Gamma$  contains no cycles).

In Section 2, we introduce a directed graph  $\Gamma(S)$  for a semigroup  $S$  with 0. We show that  $\Gamma(S)$  is not necessarily connected. Then we find a necessary and sufficient condition for  $\Gamma(S)$  to be connected. After that we extend the annihilating-ideal graph to a (not necessarily commutative) ring. It is shown that  $\mathbb{IPO}(R)$  is a semigroup. We associate to a ring  $R$  a directed graph (denote by  $\mathbb{APOG}(R)$ ) the zero-divisor graph of  $\mathbb{IPO}(R)$ , i.e.,  $\mathbb{APOG}(R) = \Gamma(\mathbb{IPO}(R))$ . Then we show that  $R$  is an Artinian (resp., Noetherian) ring if and only if  $\mathbb{APOG}(R)$  has DCC (resp., ACC) on some subset of its vertices. In Section 3, we introduce an undirected graph  $\bar{\Gamma}(S)$  for a semigroup  $S$  with 0. We show that  $\bar{\Gamma}(S)$  is always connected and  $\text{diam}(\bar{\Gamma}(S)) \leq 3$ . Moreover, if  $\bar{\Gamma}(S)$  contains a cycle, then  $\text{gr}(\bar{\Gamma}(S)) \leq 4$ . After that we define an undirected graph which extends the annihilating-ideal graph to a not necessarily commutative ring. We associate to a ring  $R$  an undirected graph (denoted by  $\overline{\mathbb{APOG}}(R)$ ) the undirected zero-divisor graph of  $\mathbb{IPO}(R)$ , i.e.,  $\overline{\mathbb{APOG}}(R) = \bar{\Gamma}(\mathbb{IPO}(R))$ . Finally, we characterize rings whose undirected annihilating-ideal graphs are complete graphs. In Section 4, we investigate the undirected annihilating-ideal graphs of matrix rings over commutative rings. It is shown that  $\text{diam}(\overline{\mathbb{APOG}}(M_n(R))) \geq 2$  where  $n \geq 2$ . Also, we show that  $\text{diam}(\overline{\mathbb{APOG}}(M_n(R))) \geq \text{diam}(\overline{\mathbb{APOG}}(R))$ .

## 2 Directed Annihilating-Ideal Graph of a Ring

Let  $S$  be a semigroup with 0 and  $D(S)$  denote the set of one-sided zero-divisors of  $S$ . We associate to  $S$  a directed graph  $\Gamma(S)$  with vertices set  $D(S)^* = D(S) \setminus \{0\}$  and  $a \rightarrow b$  if  $ab = 0$ . In this section, we investigate the properties of  $\Gamma(S)$  and we first show the following result.

**Proposition 2.1** *Let  $R$  be a ring. Then  $\mathbb{IPO}(R)$  is a semigroup.*

**Proof.** Let  $A, B \in \mathbb{IPO}(R)$ . Then there exist left or right ideals  $I_1, J_1, I_2, J_2$  of  $R$  such that  $A = I_1 J_1$  and  $B = I_2 J_2$ . We show that  $AB = (I_1 J_1)(I_2 J_2) \in \mathbb{IPO}(R)$ .

*Case 1:*  $J_1$  is a left ideal. Then  $AB = I_1(J_1 I_2 J_2) \in \mathbb{IPO}(R)$  (as  $J_1 I_2 J_2$  is a left ideal of  $R$ ).

*Case 2:*  $J_1$  is a right ideal and either  $I_2$  is a left ideal or  $J_2$  is a right ideal. Then  $AB = (I_1 J_1)(I_2 J_2) \in \mathbb{IPO}(R)$ .

*Case 3:*  $J_1$  is a right ideal,  $I_2$  is a right ideal, and  $J_2$  is a left ideal. Then  $AB = (I_1 J_1 I_2) J_2 \in \mathbb{IPO}(R)$ .

Thus  $\mathbb{IPO}(R)$  is closed multiplicatively. Since the multiplication is associative,  $\mathbb{IPO}(R)$  is a semigroup.  $\square$

It was shown in [16, Theorem 1.2] that the zero-divisor graph of a commutative semigroup  $S$  is connected and  $\text{diam}(\Gamma(S)) \leq 3$ . In the following example we show that  $\Gamma(S)$  is not necessarily connected when  $S$  is a non-commutative semigroup.

**Example 2.2** *Let  $K$  be a field and  $V = \bigoplus_{i=1}^{\infty} K$ . Then  $R = \text{HOM}_K(V, V)$ , under the point-wise addition and the multiplication taken to be the composition of functions, is an infinite non-commutative ring with identity. Let  $\pi_1 : V \rightarrow V$  be defined by  $(a_1, a_2, \dots) \mapsto (a_1, 0, \dots)$  and  $f : V \rightarrow V$  be defined by  $(a_1, a_2, \dots) \mapsto (0, a_1, a_2, \dots)$ .*

Then  $\pi_1, f \in R$ . Note that  $(R\pi_1)(fR) = 0$ , so  $\Gamma(\mathbb{IPO}(R)) \neq \emptyset$ . However,  $\Gamma(\mathbb{IPO}(R))$  is not connected as there is no path leading from the vertex  $(fR)$  to any other vertex of  $\Gamma(\mathbb{IPO}(R))$ . This is because there exists  $g : V \rightarrow V$  given by  $(a_1, a_2, \dots) \mapsto (a_2, a_3, \dots)$  and  $g \in R$  such that  $gf = 1_R$ .  $\square$

For a semigroup  $S$ , let

$$A^l(S) = \{a \in D(S)^* : \text{there exists } b \in D(R)^* \text{ such that } ba = 0\}$$

and

$$A^r(S) = \{a \in D(S)^* : \text{there exists } b \in D(R)^* \text{ such that } ab = 0\}.$$

Next we show that  $\Gamma(S)$  is connected if and only if  $A^l(S) = A^r(S)$ . Moreover, if  $\Gamma(S)$  is connected, then  $\text{diam}(\Gamma(S)) \leq 3$ .

**Theorem 2.3** *Let  $S$  be a semigroup. Then  $\Gamma(S)$  is connected if and only if  $A^l(S) = A^r(S)$ . Moreover, if  $\Gamma(S)$  is connected, then  $\text{diam}(\Gamma(S)) \leq 3$ .*

**Proof.** Suppose that  $A^l(S) = A^r(S)$ .

Let  $a$  and  $b$  be distinct vertices of  $\Gamma(S)$ . Then  $a \neq 0$  and  $b \neq 0$ . We show that there is always a path with length at most 3 from  $a$  to  $b$ .

*Case 1:*  $ab = 0$ . Then  $a \rightarrow b$  is a desired path.

*Case 2:*  $ab \neq 0$ . Then since  $A^l(S) = A^r(S)$ , there exists  $c \in D(S) \setminus \{0\}$  such that  $ac = 0$  and  $d \in D(S) \setminus \{0\}$  such that  $db = 0$ .

*Subcase 2.1:*  $c = d$ . Then  $a \rightarrow c \rightarrow b$  is a desired path.

*Subcase 2.2:*  $c \neq d$ . If  $cd = 0$ , then  $a \rightarrow c \rightarrow d \rightarrow b$  is a desired path. If  $cd \neq 0$ , then  $a \rightarrow cd \rightarrow b$  is a desired path.

Thus  $\Gamma(S)$  is connected and  $\text{diam}(\Gamma(S)) \leq 3$ .

Conversely, if  $\Gamma(S)$  is connected, then it is easy to show that  $A^l(S) = A^r(S)$ .  $\square$

Now, we define a directed graph which extends the annihilating-ideal graph to an arbitrary ring. We associate to a ring  $R$  a directed graph (denoted by  $\mathbb{APOG}(R)$ ) the zero-divisor graph of  $\mathbb{IPO}(R)$ , i.e.,  $\mathbb{APOG}(R) = \Gamma(\mathbb{IPO}(R))$ .

**Corollary 2.4** *Let  $R$  be a ring. Then  $\mathbb{APOG}(R)$  is connected if and only if  $A^l(\mathbb{IPO}(R)) = A^r(\mathbb{IPO}(R))$ . Moreover, if  $\mathbb{APOG}(R)$  is connected, then  $\text{diam}(\mathbb{APOG}(R)) \leq 3$ .*

**Proof.** Since  $\mathbb{APOG}(R)$  is equal to  $\Gamma(\mathbb{IPO}(R))$ , it follows from Theorem 2.3 that  $\mathbb{APOG}(R)$  is a connected if and only if  $A^l(\mathbb{IPO}(R)) = A^r(\mathbb{IPO}(R))$ . Also, if  $\mathbb{APOG}(R)$  is connected, then  $\text{diam}(\mathbb{APOG}(R)) \leq 3$ .

Recall that a Duo ring is a ring in which every one-sided ideal is a two-sided ideal.

**Proposition 2.5** *Let  $R$  be an Artinian Duo ring. Then  $A^l(\mathbb{IPO}(R)) = A^r(\mathbb{IPO}(R)) = \mathbb{IPO}(R) \setminus \{0, R\}$ . Moreover,  $\mathbb{APOG}(R)$  is connected and  $\text{diam}(\mathbb{APOG}(R)) \leq 3$ .*

**Proof.** Let  $R$  be a Duo ring. Then by [17, Lemma 4.2],  $R = (R_1, \mathfrak{m}_1) \times (R_2, \mathfrak{m}_2) \times \cdots (R_n, \mathfrak{m}_n)$ , where each  $R_i (1 \leq i \leq n)$  is an Artinian local ring with unique maximal ideal  $\mathfrak{m}_i$ . Let  $A \in \mathbb{IPO}(R) \setminus \{0, R\}$ . Then  $A = (I_1 \times I_2 \times \cdots \times I_n) (J_1 \times J_2 \times \cdots \times J_n)$ , where every  $I_i (1 \leq i \leq n)$  is an one-sided ideal, so is every  $J_j (1 \leq j \leq n)$ . Since  $A \neq R$ , there exists  $I_i$  (or  $J_j$ ) such that  $I_i \neq R$  (or  $J_j \neq R$ ). Without loss of generality we may assume that  $I_i \neq R$ . So  $A = (I_1 \times I_2 \times \cdots \times I_n) (J_1 \times J_2 \times \cdots \times J_n) \subseteq (R_1 \times \cdots \times I_i \times \cdots \times R_n) (R_1 \times \cdots \times R_i \times \cdots \times R_n)$ . Suppose  $k$  is the smallest positive integer such that  $I_i^k = 0$ . Thus  $(0 \times \cdots \times I_i^{k-1} \times \cdots \times 0)((R_1 \times \cdots \times I_i \times \cdots \times R_n)(R_1 \times \cdots \times R_i \times \cdots \times R_n)) = 0$ .

$\cdots \times R_i \times \cdots \times R_n)) = 0$  and  $((R_1 \times \cdots \times I_i \times \cdots \times R_n)(R_1 \times \cdots \times R_i \times \cdots \times R_n))(0 \times \cdots \times I_i^{k-1} \times \cdots \times 0) = 0$ . Therefore  $A \in A^l(\mathbb{IPO}(R))$  and  $A \in A^r(\mathbb{IPO}(R))$ . Thus  $\mathbb{IPO}(R) \setminus \{0, R\} \subseteq A^r(\mathbb{IPO}(R))$  and  $\mathbb{IPO}(R) \setminus \{0, R\} \subseteq A^l(\mathbb{IPO}(R))$ . We conclude that  $A^r(\mathbb{IPO}(R)) = \mathbb{IPO}(R) \setminus \{0, R\} = A^l(\mathbb{IPO}(R))$ .

The second part follows from Theorem 2.3.  $\square$

It is well known that if  $|D(R)| \geq 2$  is finite, then  $|R|$  is finite. Let  $A, B$  be vertices of  $\mathbb{APOG}(R)$ . We use  $A \rightleftharpoons B$  if  $A \rightarrow B$  or  $A \leftarrow B$ . For any vertices  $C$  and  $D$  of  $\mathbb{APOG}(R)$ , let  $\text{ad}(C) = \{A \text{ is a vertex of } \mathbb{APOG}(R) : C = A \text{ or } C \rightleftharpoons A \text{ or there exists a vertex } B \text{ of } \mathbb{APOG}(R) \text{ such that } C \rightleftharpoons B \rightleftharpoons A\}$  and  $\text{adu}(D) = \bigcup_{C \subseteq D} \text{ad}(C)$ . We know that  $\text{ad}(C) \subseteq D(R)$ . The following proposition shows that if a principal left or right ideal  $I$  of  $R$  is a vertex of  $\mathbb{APOG}(R)$  and all left and right ideals of  $\text{ad}(I)$  have finite cardinality, then  $R$  has finite cardinality.

**Proposition 2.6** *Let  $R$  be a ring and  $I$  be a principal left or right ideal of  $R$  such that  $I$  is a vertex of  $\mathbb{APOG}(R)$ . If all left and right ideals of  $\text{ad}(I)$  have finite cardinality, then  $R$  has finite cardinality.*

**Proof.** Without loss of generality, we may assume that  $I$  is a left principal ideal. Thus  $I = Rx$  for some non-zero  $x \in R$ . If  $\text{Ann}_l(x) = 0$ , then  $|R| = |I| < \infty$ . So we may always assume that  $\text{Ann}_l(x) \neq 0$ .

*Case 1:  $I = \text{Ann}_r(x)$  and  $\text{Ann}_r(x)\text{Ann}_l(x) = 0$ . Then*

$$I \rightarrow \text{Ann}_l(x)$$

and so  $\text{Ann}_l(x) \in \text{ad}(I)$ . Therefore,  $\text{Ann}_l(x)$  is finite. Since  $I \cong R/\text{Ann}_l(x)$ ,  $|R| = |I||\text{Ann}_l(x)| < \infty$ .

*Case 2:  $I \neq \text{Ann}_r(x)$  and  $\text{Ann}_r(x)\text{Ann}_l(x) = 0$ . If  $\text{Ann}_r(x) \neq 0$ , then*

$$I \rightarrow \text{Ann}_r(x) \rightarrow \text{Ann}_l(x)$$

and so  $\text{Ann}_l(x) \in \text{ad}(I)$ . Therefore,  $\text{Ann}_l(x)$  is finite. Since  $I \cong R/\text{Ann}_l(x)$ ,  $|R| = |I||\text{Ann}_l(x)| < \infty$ . If  $\text{Ann}_r(x) = 0$ , then since  $Rx$  is a vertex of  $\mathbb{APOG}(R)$ , there exists a (nonzero right ideal)  $J$  such that  $JRx = 0$  (replace  $J$  by  $JR$  if necessary). Since  $\text{Ann}_r(x) = 0$ , we have  $xJ$  is a nonzero right ideal and so

$$\text{Ann}_l(x) \rightarrow xJ \rightarrow I.$$

Thus  $\text{Ann}_l(x) \in \text{ad}(I)$ , so  $\text{Ann}_l(x)$  is finite. Again, we have  $|R| = |I||\text{Ann}_l(x)| < \infty$ .

*Case 3:  $I \neq \text{Ann}_r(x)$  and  $\text{Ann}_r(x)\text{Ann}_l(x) \neq 0$ . Then*

$$\text{Ann}_r(x) \leftarrow I \rightarrow \text{Ann}_r(x)\text{Ann}_l(x) \rightarrow (xR)$$

and so  $(xR), \text{Ann}_r(x) \in \text{ad}(I)$ . Therefore,  $(xR)$  and  $\text{Ann}_r(x)$  are finite. Since  $(xR) \cong R/\text{Ann}_r(x)$ ,  $|R| = |(xR)||\text{Ann}_r(x)| < \infty$ . This completes the proof.  $\square$

Here is our main result in this section.

**Theorem 2.7** *Let  $R$  be a ring such that  $\mathbb{APOG}(R) \neq \emptyset$ . Then  $R$  is Artinian (resp., Noetherian) if and only if for a left or right ideal  $I$  in the vertex set of  $\mathbb{APOG}(R)$ ,  $\text{adu}(I)$  has DCC (resp., ACC) on both its left and right ideals.*

**Proof.** If  $R$  is Artinian, then  $\mathbb{IPO}(R)$  has DCC on both its left ideals and right ideals. Thus for every left or right ideal of the vertex set of  $\mathbb{APOG}(R)$ ,  $\text{adu}(I)$  has DCC on both its left and right ideals as  $\text{adu}(I) \subseteq \mathbb{IPO}(R)$ .

Conversely, without loss of generality let  $I$  be a left ideal of vertex set of  $\mathbb{APOG}(R)$  such that  $\text{adu}(I)$  has DCC on its left and right ideals. Assume that  $x \in I$ . We have the following cases:

*Case 1:  $xRx \neq \{0\}$ ,  $\text{Ann}_l(x) \neq 0$ , and  $\text{Ann}_r(x) \neq 0$ . Then*

$$(xR) \leftarrow \text{Ann}_l(x) \leftarrow xRx \rightarrow \text{Ann}_r(x) \leftarrow (Rx).$$

Therefore  $(xR), Ann_r(x), Ann_l(x), (Rx) \in \text{ad}(xRx)$ . Since  $\text{ad}(xRx) \subseteq \text{adu}(I)$  and  $\text{adu}(I)$  has DCC on its left and right ideals, we conclude that  $(Rx)$  and  $Ann_l(x)$  are left Artinian  $R$ -modules, and  $(xR)$  and  $Ann_r(x)$  are right Artinian  $R$ -modules. Since  $(Rx) \cong R/Ann_l(x)$  and  $(xR) \cong R/Ann_r(x)$ , by [18, (1.20)] we conclude that  $R$  is Artinian.

*Case 2:*  $xRx = \{0\}$ ,  $Ann_l(x) \neq 0$ , and  $Ann_r(x) \neq 0$ . Then

$$Ann_l(x) \rightarrow (xR) \rightarrow (Rx) \rightarrow Ann_r(x).$$

Since  $\text{ad}(Rx) \subseteq \text{adu}(I)$  and  $\text{adu}(I)$  has DCC on its left and right ideals, we conclude that  $(Rx)$  and  $Ann_l(x)$  are left Artinian  $R$ -modules, and  $(xR)$  and  $Ann_r(x)$  are right Artinian  $R$ -modules. Since  $(Rx) \cong R/Ann_l(x)$  and  $(xR) \cong R/Ann_r(x)$ , by [18, (1.20)] we conclude that  $R$  is Artinian.

*Case 3:*  $Ann_l(x) = \{0\}$ . Then  $Rx \cong R$ . Therefore,  $R$  is a left Artinian module. Since  $Rx$  is a vertex of  $\text{APOG}(R)$ , we have  $Ann_r(x) \neq \{0\}$ . So there exists  $y \in D(R) \setminus \{0\}$  such that  $xy = 0$ .

*Subcase 3.1:*  $yRy \neq \{0\}$ . If  $Ann_r(y) = \{0\}$ , then since

$$Rx \rightarrow yR,$$

we have  $yR \in \text{adu}(I)$ , so  $yR$  is a Artinian right  $R$ -module. Note that  $yR \cong R$ . Therefore,  $R$  is a right Artinian module. If  $Ann_r(y) \neq \{0\}$ , then

$$Ann_r(y) \leftarrow yRy \leftarrow yRx \rightarrow yR.$$

Therefore  $(yR), Ann_r(y) \in \text{ad}(yRx) \subseteq \text{adu}(I)$ . Since  $\text{adu}(I)$  has DCC on its right ideals, we conclude that  $(yR)$  and  $Ann_r(y)$  are right Artinian  $R$ -modules. Note that  $(yR) \cong R/Ann_r(y)$ , by [18, (1.20)] we conclude that  $R$  is a right Artinian module.

*Subcase 3.2:*  $yRy = \{0\}$ . Then

$$yR \leftarrow yRx \leftarrow Ry \rightarrow Ann_r(y).$$

Since  $(yR), Ann_r(y) \in \text{ad}(yRx) \subseteq \text{adu}(I)$ , we conclude that  $(yR)$  and  $Ann_r(y)$  are right Artinian  $R$ -modules. Note that  $(yR) \cong R/Ann_r(y)$ , by [18, (1.20)] we conclude that  $R$  is a right Artinian module.

*Case 4:*  $Ann_r(x) = \{0\}$ . Then  $xRx \neq \{0\}$  and since  $Rx$  is a vertex of  $\text{APOG}(R)$ , we have  $Ann_l(x) \neq \{0\}$ . Therefore,

$$(xR) \leftarrow Ann_l(x) \rightarrow xRx.$$

We conclude that  $xR, Ann_l(x) \in \text{ad}(xRx) \subseteq \text{adu}(I)$ . Since  $xR, Rx, Ann_l(x) \in \text{adu}(I)$ , we have  $Rx$  and  $Ann_l(x)$  are left Artinian modules and  $xR$  is a right Artinian module. Note that  $(Rx) \cong R/Ann_l(x)$  and  $(xR) \cong R/Ann_r(x)$ . Again by [18, (1.20)] we conclude that  $R$  is Artinian.  $\square$

**Corollary 2.8** *Let  $R$  be a ring such that  $\text{APOG}(R) \neq \emptyset$ . Then  $R$  is Artinian (resp., Noetherian) if and only if  $\text{APOG}(R)$  has DCC (resp., ACC) on left and right ideals of its vertex set.*

**Proof.** Since vertex set of  $\text{APOG}(R)$  is a subset of  $\text{IPO}(R)$ , As in the proof of Theorem 2.7, if  $R$  is Artinian (resp., Noetherian), then  $\text{APOG}(R)$  has DCC (resp., ACC) on left and right ideals of its vertex set.

Conversely, since for a left or right ideal  $I$  of the vertex set of  $\text{APOG}(R)$ ,  $\text{adu}(I)$  is a subset of the vertex set of  $\text{APOG}(R)$ , it follows from Theorem 2.7 that  $R$  is Artinian.  $\square$

A directed graph  $\Gamma$  is called a tournament if for every two distinct vertices  $x$  and  $y$  of  $\Gamma$  exactly one of  $xy$  and  $yx$  is an edge of  $\Gamma$ . In other words, a tournament is a complete graph with exactly one direction assigned to each edge.

**Proposition 2.9** *Let  $R$  be a ring such that  $A^2 \neq \{0\}$  for every non-zero  $A \in \mathbb{IPO}(R)$  and  $A^l(\mathbb{IPO}(R)) \cap A^r(\mathbb{IPO}(R)) \neq \emptyset$ . Then  $\mathbb{APOG}(R)$  is not a tournament.*

**Proof.** Assume  $\mathbb{APOG}(R)$  is a tournament. Since  $A^l(\mathbb{IPO}(R)) \cap A^r(\mathbb{IPO}(R)) \neq \emptyset$ , there exists  $B \in A^l(\mathbb{IPO}(R)) \cap A^r(\mathbb{IPO}(R))$ , that is, there exist distinct non-zero  $A, C \in \mathbb{IPO}(R)$  such that  $A \rightarrow B \rightarrow C$  is a path in  $\mathbb{APOG}(R)$ . If  $CA \neq \{0\}$ , then  $B(CA) = (BC)A = \{0\}$  and  $(CA)B = C(AB) = \{0\}$ , which is a contradiction. So  $CA = \{0\}$  and therefore  $AC \neq \{0\}$  since  $\mathbb{APOG}(R)$  is a tournament. Also,  $AC \neq A$  (otherwise  $A^2 = (ACAC) = A(CA)C = \{0\}$ ) and similarly,  $AC \neq C$ . Let  $a, a_1 \in A$  and  $c, c_1 \in C$ . Then we have  $B \rightarrow C \rightarrow ((a - a_1c)R)$  and  $(R(c - ac_1)) \rightarrow A \rightarrow B$ . As the above  $((a - a_1c)R)B = \{0\}$  and  $B(R(c - ac_1)) = \{0\}$ . Let  $b \in B$  be an arbitrary element. Then  $-acb = a_1b - acb \in ((a - a_1c)R)B = \{0\}$  and  $bac = bc_1 - bac \in B(R(c - ac_1)) = \{0\}$ . Therefore,  $ACB = \{0\}$  and  $BAC = \{0\}$ . Thus both  $AC \rightarrow B$  and  $B \rightarrow AC$  are edges of  $\mathbb{APOG}(R)$ . This is a contradiction, hence,  $\mathbb{APOG}(R)$  cannot be a tournament.  $\square$

### 3 Undirected Annihilating-Ideal Graph of a Ring

Let  $S$  be a semigroup with 0 and recall that  $D(S)$  denotes the set of one-sided zero-divisors of  $S$ . We associate to  $S$  an undirected graph  $\overline{\Gamma}(S)$  with vertices set  $D(S)^* = D(S) \setminus \{0\}$  and two distinct vertices  $a$  and  $b$  are adjacent if  $ab = 0$  or  $ba = 0$ . Similarly, we associate to a ring  $R$  an undirected graph (denoted by  $\overline{\mathbb{APOG}}(R)$ ) the undirected zero-divisor graph of  $\mathbb{IPO}(R)$ , i.e.,  $\overline{\mathbb{APOG}}(R) = \overline{\Gamma}(\mathbb{IPO}(R))$ . The only difference between  $\mathbb{APOG}(R)$  and  $\overline{\mathbb{APOG}}(R)$  is that the former is a directed graph and the latter is undirected (that is, these graphs share the same vertices and the same edges if directions on the edges are ignored). If  $R$  is a commutative ring, this definition agrees with the previous definition of the annihilating-ideal graph. In this section we study the properties of  $\overline{\Gamma}(R)$ . We first show that  $\overline{\Gamma}(R)$  is always connected with diameter at most 3.

**Theorem 3.1** *Let  $S$  be a semigroup. Then  $\overline{\Gamma}(S)$  is a connected graph and  $\text{diam}(\overline{\Gamma}(S)) \leq 3$ .*

**Proof.** Let  $a$  and  $b$  be distinct vertices of  $\overline{\Gamma}(S)$ . If  $ab = 0$  or  $ba = 0$ , then  $a - b$  is a path. Next assume that  $ab \neq 0$  and  $ba \neq 0$ .

*Case 1:*  $a^2 = 0$  and  $b^2 = 0$ . Then  $a - ab - b$  is a path.

*Case 2:*  $a^2 = 0$  and  $b^2 \neq 0$ . Then there is some  $c \in D(S) \setminus \{a, b, 0\}$  such that either  $cb = 0$  or  $bc = 0$ . If either  $ac = 0$  or  $ca = 0$ , then  $a - c - b$  is a path. If  $ac \neq 0$  and  $ca \neq 0$ , then  $a - ca - b$  is a path if  $bc = 0$  and  $a - ac - b$  is a path if  $cb = 0$ .

*Case 3:*  $a^2 \neq 0$  and  $b^2 = 0$ . We can use an argument similar to that of the above case to obtain a path.

*Case 4:*  $a^2 \neq 0$  and  $b^2 \neq 0$ . Then there exist  $c, d \in D(S) \setminus \{a, b, 0\}$  such that either  $ca = 0$  or  $ac = 0$  and either  $db = 0$  or  $bd = 0$ . If  $bc = 0$  or  $cb = 0$ , then  $a - c - b$  is a path. Similarly, if  $ad = 0$  or  $da = 0$ ,  $a - d - b$  is a path. So we may assume that  $c \neq d$ . If  $cd = 0$  or  $dc = 0$ , then  $a - c - d - b$  is a path. Thus we may further assume that  $cd \neq 0, dc \neq 0, bc \neq 0, cb \neq 0, ad \neq 0$  and  $da \neq 0$ . We divide the proof into 4 subcases.

*Subcase 4.1:*  $ac = 0$  and  $db = 0$ . Then  $a - cd - b$  is a path.

*Subcase 4.2:*  $ac = 0$  and  $bd = 0$ . Then  $a - cb - d - b$  is a path.

*Subcase 4.3:*  $ca = 0$  and  $bd = 0$ . Then  $a - dc - b$  is a path.

*Subcase 4.4:*  $ca = 0$  and  $db = 0$ .  $a - bc - d - b$  is a path.

Thus  $\overline{\Gamma}(S)$  is connected and  $\text{diam}(\overline{\Gamma}(S)) \leq 3$ .  $\square$

In [9], Anderson and Livingston proved that if  $\Gamma(R)$  (the zero-divisor graph of a commutative ring  $R$ ) contains a cycle, then  $\text{gr}(\Gamma(R)) \leq 7$ . They also proved that  $\text{gr}(\Gamma(R)) \leq 4$  when  $R$  is Artinian and conjectured that this is the case for all commutative rings  $R$ . Their conjecture was proved independently by Mulay [19] and DeMeyer and

Schneider [15]. Also, in [20], Redmond proved that if  $\overline{\Gamma}(R)$  (the undirected zero-divisor graph of a non-commutative ring) contains a cycle, then  $\text{gr}(\overline{\Gamma}(R)) \leq 4$ . The following is our first main result in this section which shows that for a (not necessarily commutative) semigroup  $S$ , if  $\overline{\Gamma}(S)$  contains a cycle, then  $\text{gr}(\overline{\Gamma}(S)) \leq 4$ .

**Theorem 3.2** *Let  $S$  be a semigroup. If  $\overline{\Gamma}(S)$  contains a cycle, then  $\text{gr}(\overline{\Gamma}(S)) \leq 4$ .*

**Proof.** Let  $a_1 - a_2 - \cdots - a_{n-1} - a_n - a_1$  be a cycle of shortest length in  $\overline{\Gamma}(S)$ . Assume that  $\text{gr}(\overline{\Gamma}(S)) > 4$ , i.e., assume  $n \geq 5$ . Note that  $a_2 a_{n-1} \neq 0$  and  $a_{n-1} a_2 \neq 0$  (as  $n \geq 5$ ). If  $a_2 a_{n-1} \notin \{a_1, a_n\}$ , then  $a_1 - a_2 a_{n-1} - a_n - a_1$  is a cycle of length 3, yielding a contradiction. Also, if  $a_{n-1} a_2 \notin \{a_1, a_n\}$ , then  $a_1 - a_{n-1} a_2 - a_n - a_1$  is a cycle of length 3, yielding a contradiction. We have the following cases:

*Case 1 :*  $a_2 a_{n-1} = a_1$  and  $a_{n-1} a_2 = a_n$ . If  $a_2 a_3 = 0$ , then  $a_n a_3 = (a_{n-1} a_2) a_3 = 0$ . Therefore,  $a_1 - a_2 - a_3 - a_n - a_1$  is a cycle of length 4, yielding a contradiction. So,  $a_3 a_2 = 0$ . Thus,  $a_3 a_1 = a_3 (a_2 a_{n-1}) = 0$ . Therefore,  $a_1 - a_3 - a_4 - \cdots - a_{n-1} - a_n - a_1$  is a cycle of length  $n - 1$ , yielding a contradiction.

*Case 2 :*  $a_2 a_{n-1} = a_1$  and  $a_{n-1} a_2 = a_1$ . If  $a_2 a_3 = 0$ , then  $a_1 a_3 = (a_{n-1} a_2) a_3 = 0$ . Therefore,  $a_1 - a_3 - a_4 - \cdots - a_{n-1} - a_n - a_1$  is a cycle of length  $n - 1$ , yielding a contradiction. So,  $a_3 a_2 = 0$ . Thus,  $a_3 a_1 = a_3 (a_2 a_{n-1}) = 0$ . Therefore,  $a_1 - a_3 - a_4 - \cdots - a_{n-1} - a_n - a_1$  is a cycle of length  $n - 1$ , yielding a contradiction.

*Case 3 :*  $a_2 a_{n-1} = a_n$  and  $a_{n-1} a_2 = a_1$ . If  $a_2 a_3 = 0$ , then  $a_1 a_3 = (a_{n-1} a_2) a_3 = 0$ . Therefore,  $a_1 - a_3 - a_4 - \cdots - a_{n-1} - a_n - a_1$  is a cycle of length  $n - 1$ , yielding a contradiction. So,  $a_3 a_2 = 0$ . Thus,  $a_3 a_n = a_3 (a_2 a_{n-1}) = 0$ . Therefore,  $a_1 - a_2 - a_3 - a_n - a_1$  is a cycle of length 4, yielding a contradiction.

*Case 4 :*  $a_2 a_{n-1} = a_n$  and  $a_{n-1} a_2 = a_n$ . If  $a_2 a_3 = 0$ , then  $a_n a_3 = (a_{n-1} a_2) a_3 = 0$ . If  $a_3 a_2 = 0$ , then  $a_3 a_n = a_3 (a_2 a_{n-1}) = 0$ . Therefore,  $a_1 - a_2 - a_3 - a_n - a_1$  is a cycle of length 4, yielding a contradiction.

Since in all cases we have found contradictions, we conclude that if  $\overline{\Gamma}(S)$  contains a cycle, then  $\text{gr}(\overline{\Gamma}(S)) \leq 4$ .

□

**Corollary 3.3** *Let  $R$  be a ring. Then  $\overline{\text{APOG}}(R)$  is a connected graph and  $\text{diam}(\overline{\text{APOG}}(R)) \leq 3$ . Moreover, If  $\overline{\text{APOG}}(R)$  contains a cycle, then  $\text{gr}(\overline{\text{APOG}}(R)) \leq 4$ .*

**Proof.** Note that  $\overline{\text{APOG}}(R)$  is equal to  $\overline{\Gamma}(\text{IPO}(R))$ . So by Theorem 3.1,  $\overline{\text{APOG}}(R)$  is a connected graph and  $\text{diam}(\overline{\text{APOG}}(R)) \leq 3$ . Also, by Theorem 3.2, if  $\overline{\text{APOG}}(R)$  contains a cycle, then  $\text{gr}(\overline{\text{APOG}}(R)) \leq 4$ . □

For a not necessarily commutative ring  $R$ , we define a simple undirected graph  $\overline{\Gamma}(R)$  with vertex set  $D(R)^*$  (the set of all non-zero zero-divisors of  $R$ ) in which two distinct vertices  $x$  and  $y$  are adjacent if and only if either  $xy = 0$  or  $yx = 0$  (see [20]). The Jacobson radical of  $R$ , denoted by  $J(R)$ , is equal to the intersection of all maximal right ideals of  $R$ . It is well-known that  $J(R)$  is also equal to the intersection of all maximal left ideals of  $R$ . In our second main theorem in this section we characterize rings whose undirected annihilating-ideal graphs are complete graphs.

**Theorem 3.4** *Let  $R$  be a ring. Then  $\overline{\text{APOG}}(R)$  is a complete graph if and only if either  $(D(R))^2 = 0$ , or  $R$  is a direct product of two division rings, or  $R$  is a local ring with maximal ideal  $\mathfrak{m}$  such that  $\text{IPO}(R) = \{0, \mathfrak{m}, \mathfrak{m}^2, R\}$ .*

**Proof.** Assume that  $\overline{\text{APOG}}(R)$  is a complete graph. If  $\overline{\Gamma}(R)$  is a complete graph, then by [6, Theorem 5], either  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  or  $D(R)^2 = \{0\}$ . So the forward direction holds. Next assume that  $\overline{\Gamma}(R)$  is not a complete graph. So there exist different vertices  $x$  and  $y$  of  $\overline{\Gamma}(R)$  such that  $x$  and  $y$  are not adjacent. We have the following cases:

*Case 1:*  $x \in A^r(R)$ . Without loss of generality assume that  $y \in A^r(R)$ . If  $Rx \neq Ry$ , then since  $\overline{\text{APOG}}(R)$  is a complete graph, we have  $Rx$  is adjacent to  $Ry$  in  $\overline{\text{APOG}}(R)$ , so  $x$  and  $y$  are adjacent in  $\overline{\Gamma}(R)$ , yielding a contradiction. Thus  $Rx = Ry$ . Since  $x \in A^r(R)$ , there exists non-zero element  $z \in D(R)$  such that  $xz = 0$ . If  $Rx \subseteq zR$ , then  $(Rx)^2 = \{0\}$ . So  $(Rx)(Ry) = \{0\}$ , and  $x$  and  $y$  are adjacent in  $\overline{\Gamma}(R)$ , yielding a contradiction. Therefore,  $Rx \not\subseteq zR$ . If there exists a left or right ideal  $I$  of  $R$  except  $zR$  such that  $I \not\subseteq Rx$ , then there exists

nonzero element  $s \in I \setminus Rx$ . Then  $(Rs + Rx)(zR) = \{0\}$ . Since  $\overline{\text{APOG}}(R)$  is a complete graph  $Rx$  is adjacent to  $(Rs + Rx) = \{0\}$ . Thus  $(Rx)^2 = \{0\}$ , and so  $x$  and  $y$  are adjacent in  $\overline{\Gamma}(R)$ , yielding a contradiction. Therefore,  $\{zR, Rx\}$  is the set of nonzero proper left or right ideals of  $R$ . Thus by Corollary 2.8,  $R$  is an Artinian ring. We have the following subcases:

*Subcase 1:*  $zR \not\subseteq Rx$ . Then  $zR$  and  $Rx$  are maximal ideals. If  $zR$  or  $Rx$  is not a two-sided ideal, then  $zR = J(R) = Rx$ , yielding a contradiction. Therefore,  $Rx$  and  $zR$  are two-sided ideals. Also,  $Rx$  and  $zR$  are minimal ideals and so  $Rx \cap zR = \{0\}$ . Thus by Brauer's Lemma (see [18, 10.22]),  $(Rx)^2 = 0$  or  $Rx = Re$ , where  $e$  is a idempotent in  $R$ . If  $(Rx)^2 = \{0\}$ , then  $x$  is adjacent to  $y$  in  $\overline{\Gamma}(R)$ , yielding a contradiction. So  $Rx = Re$ , where  $e$  is an idempotent in  $R$ . Therefore,  $R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$ . Since  $\{zR, Rx\}$  is the set of nonzero proper left or right ideals of  $R$  and  $Rx \cap zR = \{0\}$ , we conclude that  $Re = Rx = eR$  and  $(1-e)R = zR = R(1-e)$ . Therefore,  $(1-e)Re = (1-e)eR = \{0\}$  and  $eR(1-e) = e(1-e)R = \{0\}$ . So  $R = eRe \oplus (1-e)R(1-e)$ . Since  $R$  is an Artinian ring with two nonzero left or right ideals, we conclude that  $eRe$  and  $(1-e)R(1-e)$  are division rings.

*Subcase 2:*  $zR \subseteq Rx$ . Then  $Rx = D(R)$ . If  $(Rx)^2 = \{0\}$ , then  $x$  is adjacent to  $y$  in  $\overline{\Gamma}(R)$ , yielding a contradiction. If  $D(R)^2 \neq 0$ , then  $D(R)^2 = zR$ . Therefore,  $R$  is a local ring with maximal ideal  $\mathfrak{m}$  such that  $\text{IPO}(R) = \{0, \mathfrak{m}, \mathfrak{m}^2, R\}$ .

In summary, we obtain that either  $R$  is a direct product of two division rings, or  $R$  is a local ring with maximal ideal  $\mathfrak{m}$  such that  $\text{IPO}(R) = \{0, \mathfrak{m}, \mathfrak{m}^2, R\}$ . Thus the forward direction holds.

*Case 2:*  $x \in A^l(R)$ . Similar to Case 1, we conclude that either  $R$  is a direct product of two division rings, or  $R$  is a local ring with maximal ideal  $\mathfrak{m}$  such that  $\text{IPO}(R) = \{0, \mathfrak{m}, \mathfrak{m}^2, R\}$ . So the forward direction holds.

The converse is obvious. □

## 4 Undirected Annihilating-Ideal Graphs for Matrix Rings Over Commutative Rings

In this section we investigate the undirected annihilating-ideal graphs of matrix rings over commutative rings. By Theorem 3.3,  $\text{diam}(\overline{\text{APOG}}(R)) \leq 3$  for any ring  $R$ . In Proposition 4.1 we show that  $\text{diam}(\overline{\text{APOG}}(M_n(R))) \geq 2$  where  $n \geq 2$ . A natural question is whether or not  $\text{diam}(\overline{\text{APOG}}(M_n(R))) \geq \text{diam}(\overline{\text{APOG}}(R))$ . We show that the answer to this question is affirmative.

**Proposition 4.1** *Let  $R$  be a commutative ring. Then  $\text{diam}(\overline{\text{APOG}}(M_n(R))) \geq 2$  where  $n \geq 2$ .*

**Proof.** Let

$$A = (M_n(R) \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}) \text{ and } B = ( \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} M_n(R)).$$

Since

$$A \left( \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} M_n(R) \right) = 0 \text{ and } (M_n(R) \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}) B = 0,$$



we conclude that  $A$  and  $B$  are vertices in  $(\overline{\text{APOG}}(M_n(R)))$ . Note that

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^2 \neq 0 \text{ and } \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in A \cap B,$$

so  $AB \neq 0$ . Therefore,  $\text{diam}(\overline{\text{APOG}}(M_n(R))) \geq 2$ .  $\square$

**Theorem 4.2** *Let  $R$  be a commutative ring. Then  $\text{diam}(\overline{\text{APOG}}(M_n(R))) \geq \text{diam}(\text{AG}(R)) = \text{diam}(\overline{\text{APOG}}(R))$ .*

**Proof.** By [12, Theorem 2.1],  $\text{diam}(\text{AG}(R)) \leq 3$ .

*Case 1:*  $\text{diam}(\text{AG}(R)) \leq 2$ . By Proposition 4.1,  $\text{diam}(\overline{\text{APOG}}(M_n(R))) \geq 2$ . Thus  $\text{diam}(\overline{\text{APOG}}(M_n(R))) \geq \text{diam}(\text{AG}(R))$ .

*Case 2:*  $\text{diam}(\text{AG}(R)) = 3$ . Then there exist vertices  $I, J, K$ , and  $L$  of  $\text{AG}(R)$  such that  $I - K - L - J$  is a shortest path between  $I$  and  $J$ . So  $d(I, J) = 3$ . Since  $I$  and  $J$  are vertices of  $\text{AG}(R)$ ,  $M_n(I)$  and  $M_n(J)$  are vertices of  $\overline{\text{APOG}}(M_n(R))$ . Suppose that  $\text{diam}(\overline{\text{APOG}}(M_n(R))) = 2$ . So we can assume that there exists  $\alpha = [a_{ij}] \in M_n(R)$  such that  $M_n(I)\alpha = \alpha M_n(J) = 0$ . Without loss of generality, we may assume that  $a_{11} \neq 0$ . For every  $a \in I$ ,

$$\begin{bmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} A = 0,$$

so  $aa_{11} = 0$ . Therefore  $I(a_{11}R) = 0$ . For every  $b \in J$ ,

$$A \begin{bmatrix} b & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = 0.$$

Therefore  $(a_{11}R)J = 0$ . Thus  $I - (a_{11}R) - J$  is a path of length 2 in  $\text{AG}(R)$ , and so  $d(I, J) \leq 2$ , yielding a contradiction. Therefore,  $\text{diam}(\overline{\text{APOG}}(M_n(R))) = 3$  and we are done.  $\square$

It was shown in Corollary 3.3 that  $\text{gr}(\overline{\text{APOG}}(R)) \leq 4$ . We now show that  $\text{gr}(\overline{\text{APOG}}(M_n(R))) = 3$  where  $n \geq 2$ .

**Proposition 4.3** *Let  $R$  be a commutative ring. Then  $\text{gr}(\overline{\text{APOG}}(M_n(R))) = 3$  where  $n \geq 2$ .*

**Proof.** Let

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then  $(AM_n(R)A) - (BM_n(R)B) - (CM_n(R)C)$  is a cycle in  $(\overline{\text{APOG}}(M_n(R)))$ , so  $\text{gr}(\overline{\text{APOG}}(M_n(R))) = 3$ .  $\square$

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